Announcements

1) Typos fixed on HW #3:
square added on 2),
indexing fixed on 3),
turn in Monday

2) Fix to notes - complex
inner product from a norm
with the parallelogram property

3) Colloquium tomorrow
3-4 CB 2070
inscribed ellipses in
n-gons ($n = 3, 4, 5$)

4) Midterm (in-class)
Tuesday 10/22

Definition: (orthogonality)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with an inner product $\langle \cdot , \cdot \rangle$ (V is an inner-product space).

Two vectors x and y in V are orthogonal

if $\boxed{\langle x, y \rangle = 0}$.

Geometrically, this means x any y are perpendicular in the plane they determine - provided neither is 0_V!

We say a set of vectors is orthogonal if $\langle x,y \rangle = 0$ $\forall$ x,y in the set, $x \neq y$ (unless $x = 0_V$).

Theorem (orthogonality and linear independence)

Let V be an inner product space and let $S \subseteq V$ be a set of nonzero, orthogonal vectors.

Then S is a linearly independent subset of V.

Proof: Let S be a set of nonzero orthogonal vectors contained in V.

Let $v_1, v_2, \ldots, v_n \in V$ with $v_i \neq v_j$ for $i \neq j$, and let $\alpha_1, \ldots, \alpha_n$ be scalars.

Suppose

$$\sum_{i=1}^{n} \alpha_i v_i = 0_v$$

(i.e. $\{v_1, \ldots, v_n\}$ is linearly dependent)

Fix j, $1 \leq j \leq n$.

Consider

$$\left\langle \sum_{i=1}^{n} \alpha_i v_i , v_j \right\rangle$$

$$= \sum_{i=1}^{n} \alpha_i \langle v_i , v_j \rangle$$

by linearity in the first coordinate

$= \alpha_j \langle v_j , v_j \rangle$ since $\langle v_i , v_j \rangle = 0 \ \forall \ i \neq j$.

But we assumed

$$\sum_{i=1}^{n} \alpha_i v_i = 0_V, \text{ so}$$

$$\begin{aligned} 0 &= \langle 0_V, v_j \rangle \\ &= \left\langle \sum_{i=1}^{n} \alpha_i v_i, v_j \right\rangle \\ &= \alpha_j \langle v_j, v_j \rangle \\ &= \alpha_j \| v_j \|_2^2 . \end{aligned}$$

Since $v_j \neq 0_V$,

$\|v_j\|_2^2 > 0$, so

dividing by $\|v_j\|_2^2$

in the equality

$$0 = \alpha_j \|v_j\|_2^2,$$

we get $\alpha_j = 0$.

Since $1 \leq j \leq n$ was arbitrary,

$\alpha_j = 0 \ \forall \ 1 \leq j \leq n$, i.e.

S is linearly independent. $\square$

Definition: (normalization)

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and let $\|\cdot\|$ be a norm on V.

A vector $x \in V$ is **normalized** if $\|x\| = 1$.

If $x \neq 0_V$, then we may normalize x by taking the vector $y = \frac{x}{\|x\|}$.

Then $\|y\| = \left\| \frac{x}{\|x\|} \right\|$

$$= \frac{1}{\cancel{\|x\|}} \cancel{\|x\|}$$

$$= 1.$$

Normalizing a set of vectors leaves their span and linear independence properties unchanged.

Definition: (orthonormality)

Let V be an inner product space. A set of vectors $S \subseteq V$ is **orthonormal** (orthoganal and normalized) if

$$\langle x, y \rangle = \delta_{x,y} = \begin{cases} 1, & x = y \\ 0, & x \neq y \end{cases}$$

$\forall\ x, y \in S$.

If S is an orthonormal set that spans V, we say S is an **orthonormal basis** for V.

Example 1: In $\mathbb{R}^n$ or $\mathbb{C}^n$

with the standard inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

for $x = (x_i)_{i=1}^{n}$ and $y = (y_i)_{i=1}^{n}$,

an orthonormal basis is given by the standard basis.

Observe

$$\langle e_i, e_j \rangle = \delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and we already know

$\{e_i\}_{i=1}^{n}$ is spanning.

Example 2: In C_{00}, the space of all complex sequences that are eventually zero (as a vector space over $\mathbb{C}$), the set

$$\{e_i\}_{i=1}^{\infty}$$

is an orthonormal basis with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

for $x = (x_i)_{i=1}^{\infty}, y = (y_i)_{i=1}^{\infty} \in c_{00}$.

Note that for any two given vectors, the above sum is finite and so always makes sense.

It is immediate that $\langle e_i, e_j \rangle = \delta_{i,j}$ and we already know spanning.

Gram-Schmidt Orthogonalization Procedure (finite dimensions)

For the moment, consider

$V = \mathbb{R}^3$ over $\mathbb{R}$.

Consider the set

$\{w_1, w_2, w_3\}$ where

$$w_1 = (1, 2, 3)$$
$$w_2 = (0, 2, 3)$$
$$w_3 = (1, 4, 5).$$

This set is linearly independent but not orthogonal.

Make a set of three vectors that are orthnoganal by using $\{w_1, w_2, w_3\}$ as follows:

Let $x_1 = w_1$.

$$\langle x_1, w_2 \rangle = \langle w_1, w_2 \rangle$$
$$= 13 \neq 0.$$

Make x_2 from

w_1 and w_2 by

$$x_2 = w_2 - \frac{w_1}{\|w_1\|_2^2} \cdot \langle x_1, w_2 \rangle .$$

$$= w_2 - \frac{x_1}{\|x_1\|_2^2} \langle x_1, w_2 \rangle$$

Then

$$\langle x_1, x_2 \rangle = \left\langle x_1, w_2 - \frac{w_1}{\|w_1\|_2^2} \langle x_1, w_2 \rangle \right\rangle$$

$$= \left\langle w_1, w_2 - \frac{w_1}{\|w_1\|_2^2} \langle w_1, w_2 \rangle \right\rangle$$

$$= \langle w_1, w_2 \rangle - \left\langle w_1, \frac{w_1 \langle w_1, w_2 \rangle}{\|w_1\|_2^2} \right\rangle$$

$$= \langle w_1, w_2 \rangle - \frac{\langle w_1, w_2 \rangle \cancel{\|w_1\|_2^2}}{\cancel{\|w_1\|_2^2}}$$

$$= 0$$

How to make x_3?

$$x_3 = w_3 - \frac{x_2}{\|x_2\|_2^2}\langle w_3, x_2\rangle - \frac{x_1}{\|x_1\|_2^2}\langle w_3, x_1\rangle$$

Recall that $\langle x_1, x_2\rangle = 0$.

$$w_1 = (1,2,3)$$
$$w_2 = (0,2,3)$$
$$w_3 = (1,4,5)$$

$$x_1 = w_1$$
$$x_2 = w_2 - \frac{\langle x_1, w_2 \rangle}{\|x_1\|_2^2} x_1$$
$$= (0,2,3) - \frac{13}{14}(1,2,3)$$
$$= \left(-\frac{13}{14}, \frac{2}{14}, \frac{3}{14} \right)$$

$$x_3 = w_3 - \frac{\langle w_3, x_2\rangle}{\|x_2\|_2^2} x_2 - \frac{\langle w_3, x_1\rangle}{\|x_1\|_2^2} x_1$$

$$= (1,4,5) - \frac{10\cdot 14^{\cancel{2}}}{180}\left(-\frac{13}{\cancel{14}}, \frac{2}{\cancel{14}}, \frac{3}{\cancel{14}}\right)$$

$$- \frac{24}{14}(1,2,3)$$

$$= (1,4,5) - \frac{1}{18}(-13,2,3)$$

$$- \frac{12}{7}(1,2,3) = \left(0, {}^{6}/_{13}, {}^{-4}/_{13}\right)$$

We check:

$$\langle x_1, x_3\rangle = \langle (1,2,3), \frac{1}{13}(0,6,-4)\rangle$$

$$= \frac{12}{13} - \frac{12}{13} = 0$$

$$\langle x_2, x_3\rangle = \langle \frac{1}{14}(-13,2,3), \frac{1}{13}(0,6,-4)\rangle$$

$$= \frac{12}{182} - \frac{12}{182} = 0$$

So x_3 is indeed orthogonal to both x_1 and x_2.